

ON REDUCED ARAKELOV DIVISORS OF REAL QUADRATIC FIELDS

HA THANH NGUYEN TRAN

ABSTRACT. We generalize the concept of reduced Arakelov divisors and define C -reduced divisors for a given number $C \geq 1$. These C -reduced divisors have remarkable properties which are similar to the properties of reduced ones. In this paper, we describe an algorithm to test whether an Arakelov divisor of a real quadratic field F is C -reduced in time polynomial in $\log |\Delta_F|$ with Δ_F the discriminant of F . Moreover, we give an example of a cubic field for which our algorithm does not work.

1. INTRODUCTION

The idea of infrastructure of real quadratic fields of Shanks in [11] was modified and extended by Lenstra [5], Schoof [9] and Buchmann and Williams [2] to certain number fields. Finally, it was generalized to arbitrary number fields by Buchmann [1]. In 2008, Schoof [10] gave the first description of infrastructure in terms of reduced Arakelov divisors and the Arakelov class group Pic_F^0 of a general number field F . Reduced Arakelov divisors can be used for computing Pic_F^0 . They form a finite and regularly distributed set in this topological group [10, Proposition 7.2, Theorem 7.4 and 7.7]. Computing Pic_F^0 is of interest because knowing this group is equivalent to knowing the class group and the unit group of F (see [6] and [10]).

Schoof proposed two algorithms which run in polynomial time in $\log |\Delta_F|$ with Δ_F the discriminant of F [10, Algorithm 10.3]: the testing algorithm to check whether a given Arakelov divisor D is reduced, and the reduction algorithm to compute a reduced Arakelov divisor that is close to a given divisor D in Pic_F^0 . However, the reduction algorithm requires finding a shortest vector of the lattice associated to the Arakelov divisor, while finding a reasonably short vector using the LLL algorithm is much faster and easier than finding a shortest vector. This leads to modifications and generalizations of the definition of reduced Arakelov divisors.

One of the generalizations, which we call C -reduced Arakelov divisors, comes from the reduction algorithm of Schoof [10, Algorithm 10.3]. With this definition, C -reduced Arakelov divisors are reduced in the usual sense when $C = 1$, and Arakelov divisors that are reduced in the usual sense are C -reduced with $C = \sqrt{n}$ (see [10]). C -reduced divisors still form a finite and regularly distributed set in Pic_F^0 , just like the reduced divisors.

This modification, however, has a drawback, since for general number fields it is not known how to test whether a given divisor is C -reduced. Currently, we have a testing

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algorithm to do this only for real quadratic fields, in time polynomial in $\log(|\Delta_F|)$. It is the main result of this paper, presented in Section 4.

In Section 2, we discuss C -reduced Arakelov divisors in an arbitrary number field. Section 3 is devoted to the properties of C -reduced fractional ideals of real quadratic fields. An example of real cubic fields in which the testing algorithm is no longer efficient is given in Section 5.

2. C -REDUCED ARAKELOV DIVISORS

In this section, we introduce C -reduced Arakelov divisors of number fields.

Let F be a number field of degree n and r_1, r_2 the numbers of real and complex infinite primes (or infinite places) of F , respectively. Let

$$F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma \text{ real}} \mathbb{R} \times \prod_{\sigma \text{ complex}} \mathbb{C}.$$

Here σ 's are the infinite primes of F . Then $F_{\mathbb{R}}$ is an étale \mathbb{R} -algebra with the canonical Euclidean structure given by the scalar product

$$\langle u, v \rangle := \text{Tr}(u\bar{v}) \quad \text{for } u = (u_{\sigma})_{\sigma}, v = (v_{\sigma})_{\sigma} \in F_{\mathbb{R}}.$$

In particular, in terms of coordinates, we have

$$\|u\|^2 = \text{Tr}(u\bar{u}) = \sum_{\sigma \text{ real}} |u_{\sigma}|^2 + 2 \sum_{\sigma \text{ complex}} |u_{\sigma}|^2, \quad \text{for any } u = (u_{\sigma})_{\sigma} \in F_{\mathbb{R}}.$$

The *norm* of an element $u = (u_{\sigma})_{\sigma}$ of $F_{\mathbb{R}}$ is defined by

$$N(u) := \prod_{\sigma \text{ real}} u_{\sigma} \cdot \prod_{\sigma \text{ complex}} |u_{\sigma}|^2.$$

Definition 2.1. An *Arakelov divisor* is a formal finite sum

$$D = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p} + \sum_{\sigma} x_{\sigma} \sigma$$

where \mathfrak{p} runs over the nonzero prime ideals in O_F and σ runs over the infinite primes of F , with $n_{\mathfrak{p}} \in \mathbb{Z}$ but $x_{\sigma} \in \mathbb{R}$.

To each divisor D we associate the *Hermitian line bundle* (I, u) where $I = \prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}$ is a fractional ideal in F and $u = (e^{-x_{\sigma}})_{\sigma}$ is a vector in $\prod_{\sigma} \mathbb{R}_{>0} \subset F_{\mathbb{R}}$.

There is a natural way to associate an ideal lattice to D . Indeed, I is embedded into $F_{\mathbb{R}}$ by the infinite primes σ . Each element g of I is mapped to the vector $(\sigma(g))_{\sigma}$ in $F_{\mathbb{R}}$. Since the vector $ug := (u_{\sigma} \sigma(g))_{\sigma} \in F_{\mathbb{R}}$, we can define

$$\|g\|_D := \|ug\|.$$

In terms of coordinates, we have

$$\|g\|_D^2 = \sum_{\sigma \text{ real}} u_{\sigma}^2 |\sigma(g)|^2 + 2 \sum_{\sigma \text{ complex}} |u_{\sigma}|^2 |\sigma(g)|^2.$$

With this metric, I becomes an ideal lattice in $F_{\mathbb{R}}$. We call I *the ideal lattice associated to D* . The vector u has the role of a metric for I . Hence we make the following definition.

Definition 2.2. Let I be a fractional ideal in F and let u be in $F_{\mathbb{R}}^*$. The *length of an element g of I with respect to the metric u* is defined by $\|g\|_u := \|ug\|$.

Definition 2.3. Let I be a fractional ideal. Then 1 is called *primitive* in I if 1 belongs to I and it is not divisible by any integer ≥ 2 .

Definition 2.4. Let $C \geq 1$. A fractional ideal I is called *C -reduced* if:

- 1 is primitive in I .
- There exists a metric $u \in \prod_{\sigma} \mathbb{R}_{>0}$ such that $\|1\|_u \leq C\|g\|_u$ for all $g \in I \setminus \{0\}$.

Remark 2.5. The second condition of Definition 2.4 is equivalent to saying that there exists a metric u such that with respect to this metric, the vector 1 scaled by the scalar C is a shortest vector in the lattice I .

Definition 2.6. Let I be a fractional ideal in F . The *Arakelov divisor* $d(I)$ is defined to be associated with the Hermitian line bundle (I, u) where $u = (u_{\sigma})_{\sigma}$ with $u_{\sigma} = N(I)^{-1/n}$ for all σ .

Definition 2.7. An Arakelov divisor D is called *C -reduced* if it has the form $D = d(I)$ for some C -reduced fractional ideal I .

Now we prove the following lemma.

Lemma 2.8. *Let I be a fractional ideal. If I is C -reduced then the inverse I^{-1} of I is an integral ideal and its norm is at most $C^n \partial_F$ where $\partial_F = (2/\pi)^{r_2} \sqrt{|\Delta_F|}$.*

Proof. Since $1 \in I$, we have $I^{-1} \subset O_F$. Then $L = N(I)^{-1/n} I$ is a lattice of covolume $\sqrt{|\Delta_F|}$ [10, Section 4]. Consider the symmetric, convex and bounded subset of $F_{\mathbb{R}}$,

$$S = \{(x_{\sigma})_{\sigma} : |x_{\sigma}| < \partial_F^{1/n} \text{ for all } \sigma\}.$$

For real σ , the segment $|x_{\sigma}| < \partial_F^{1/n}$ in \mathbb{R} has length $2 \cdot \partial_F^{1/n}$. For complex σ , the disc $|x_{\sigma}| < \partial_F^{1/n}$ in \mathbb{C} has area $2\pi(\partial_F^{1/n})^2$. Thus,

$$\text{vol}(S) = (2\partial_F^{1/n})^{r_1} \cdot (2\pi(\partial_F^{1/n})^2)^{r_2} = 2^{r_1} (2\pi)^{r_2} \partial_F = 2^n \text{covol}(L).$$

By Minkowski's theorem, there is a nonzero element $f \in I$ such that

$$N(I)^{-1/n} |\sigma(f)| \leq \partial_F^{1/n} \text{ for all } \sigma.$$

Since I is C -reduced, there exists a metric u such that $\|1\|_u \leq C\|f\|_u$. This implies that $\|u\| \leq C\|u\| \max_{\sigma} |\sigma(f)| \leq C\|u\| \partial_F^{1/n} N(I)^{1/n}$. Hence $N(I^{-1}) \leq C^n \partial_F$. \square

Remark 2.9. In this paper, given a fractional ideal I , we assume that it is represented by a matrix with rational entries as in [7, Section 4] and [6, Section 2]. Without loss of generality, we can also assume that the length of the input is polynomial in $\log |\Delta_F|$.

By Lemma 2.8, to test whether I is C -reduced, first we can check that $N(I)^{-1} \leq C^n \partial_F$. We have the following.

Lemma 2.10. *Testing $N(I)^{-1} \leq C^n \partial_F$ can be done in time polynomial in $\log |\Delta_F|$.*

Proof. Let M be the matrix representation of I . Since we know that $N(I)^{-1} = \sqrt{|\Delta_F|} / \text{covol}(I)$, it is sufficient to check that

$$|\det(M)| = \text{covol}(I) > (\pi/2)^{r_2} C^n.$$

Recall that the determinant of the matrix M can be computed in time polynomial [8, Section 1]. This reason and Remark 2.9 imply that testing $N(I)^{-1} \leq C^n \partial_F$ can be done in time polynomial in $\log |\Delta_F|$. \square

Regarding the primitiveness of 1 in I , we have the result below.

Lemma 2.11. *Let $C \geq 1$ and let I be a fractional ideal containing 1 with $N(I)^{-1} \leq C^n \partial_F$. Then testing whether or not 1 is primitive can be done in time polynomial in $\log |\Delta_F|$.*

Proof. Let $\{c_1, \dots, c_n\}$ be an LLL-reduced \mathbb{Z} -basis of O_F and $\{b_1, \dots, b_n\}$ be an LLL-reduced \mathbb{Z} -basis of I^{-1} . Since $1 \in I$, we obtain $I^{-1} \subset O_F$ and so $b_i \in O_F$ for all i . Then for each $i = 1, \dots, n$, there exist the integers k_{ij} with $j = 1, \dots, n$ for which $b_i = \sum_j k_{ij} c_j$. Thus, there is an integer d such that $1/d \in I$ if and only if $I^{-1} \subset dO_F$. This is equivalent to $d | k_{ij}$ for all i, j . In other words, $d | \gcd(k_{ij}, 1 \leq i, j \leq n)$. In conclusion, 1 is primitive in I if and only if $\gcd(k_{ij}, 1 \leq i, j \leq n) = 1$.

Since $N(I)^{-1} \leq C^n \partial_F$, an LLL-reduced \mathbb{Z} -basis of I , the coefficients k_{ij} and $\gcd(k_{ij}, 1 \leq i, j \leq n)$ can be computed in polynomial time in $\log |\Delta_F|$. In other words, testing the primitiveness of 1 can be done in time polynomial in $\log |\Delta_F|$. \square

By Lemma 2.11, we know how to test the first condition of Definition 2.4. From now on, we only consider the second condition of this definition.

Remark 2.12. Note that if $u \in \prod_{\sigma} \mathbb{R}_{>0}$ satisfies the second condition of Definition 2.4, then $u' = (u_{\sigma} / N(u)^{1/n})_{\sigma} \in \prod_{\sigma} \mathbb{R}_{>0}$ still satisfies that condition and $N(u') = 1$. Therefore, we can always assume that $N(u) = 1$ from now on.

Proposition 2.13. *Let I be a fractional ideal and u be a vector satisfying the second condition of Definition 2.4 with $N(u) = 1$. Then*

$$\|u\| \leq C \sqrt{n} (2/\pi)^{r_2/n} \text{covol}(I)^{1/n}.$$

Proof. Let $L = uI := \{uf = (u_{\sigma} \cdot \sigma(f))_{\sigma} : f \in I\} \subset F_{\mathbb{R}}$. Then L is a lattice with metric inherited from $F_{\mathbb{R}}$ (see [10]). Since $N(u) = 1$, the lattice L has covolume equal to $\text{covol}(I)$. Consider the symmetric, convex and bounded subset S of $F_{\mathbb{R}}$

$$S = \{(x_{\sigma}) : |x_{\sigma}| < (2/\pi)^{r_2/n} \text{covol}(I)^{1/n} \text{ for all } \sigma\}.$$

We have

$$\text{vol}(S) = 2^{r_1} (2\pi)^{r_2} (2/\pi)^{r_2} \text{covol}(I) = 2^n \text{covol}(L).$$

By Minkowski's theorem, there is a nonzero element $f \in I$ such that

$$u_{\sigma} |\sigma(f)| \leq (2/\pi)^{r_2/n} \text{covol}(I)^{1/n} \text{ for all } \sigma.$$

So

$$\|uf\| \leq \sqrt{n}(2/\pi)^{r_2/n} \text{covol}(I)^{1/n}.$$

Because u satisfies the second condition of Definition 2.4, we have $\|u\| \leq C\|uf\|$. The proposition is then proved. \square

3. C -REDUCED ARAKELOV DIVISORS OF REAL QUADRATIC FIELDS

In this part, fix $C \geq 1$ and a real quadratic field F with the discriminant Δ_F , we will describe what C -reduced ideals look like, and we will investigate their properties.

Here and in the rest of the paper, we often identify an element g of fractional ideals with its image $(\sigma(g))_\sigma \in F_{\mathbb{R}}$. Thus, elements of fractional ideals of real quadratic fields have the form $g = (g_1, g_2) \in F_{\mathbb{R}} \cong \mathbb{R}^2$.

Remark 3.1. Let F be an imaginary quadratic field and let I be a fractional ideal of F . Then an element $g \in I$ can be identified with its image $g \in F_{\mathbb{R}} \cong \mathbb{C}$. The second condition of Definition 2.4 is equivalent to: there exists $u \in \mathbb{R}_{>0}$ such that $|u| \leq C|ug|$ for all $g \in I \setminus \{0\}$. Since u is a positive real number, this is equivalent to that $1/C \leq |g|$ for all $g \in I \setminus \{0\}$. In other words, the shortest vectors of I have length at least $1/C$. In addition, the first vector in an LLL reduced basis of I is also its shortest vector; finding this vector can also be done in polynomial time. This together with Lemma 2.11 shows that whether a given ideal of an imaginary quadratic field is C -reduced can be tested easily and in polynomial time. Therefore, in this section, we only consider C -reduced ideals of real quadratic fields.

3.1. A geometrical view of reduced ideals in real quadratic cases. We have $F_{\mathbb{R}} \cong \mathbb{R}^2$. Let I be a fractional ideal of F and S_1 be the square centered at the origin of $F_{\mathbb{R}}$ which has a vertex $(1/C, 1/C)$. We have the following result.

Proposition 3.2. *The second condition in Definition 2.4 can be restated as follows. There exists an ellipse E_4 , centered at the origin and passing through the vertices of S_1 , whose interior does not contain any nonzero points of the lattice I .*

Proof. It is easy to see by writing down the condition $\|u\| \leq C\|uf\|$ in terms of the coordinates of u and f . \square

Proposition 3.3. *If I has some nonzero element in the square S_1 then the ellipse E_4 described in Proposition 3.2 does not exist. On the other hand, E_4 exists when the shortest vectors of I have length at least $\sqrt{2}/C$.*

Proof. For the first case, we assume that there is a nonzero element g of I in the square S_1 . Since S_1 is inside E_4 , so is g (see Figure 1). In the second case, we can take for E_4 the circle E_1 centered at the origin and of radius $\sqrt{2}/C$. Because the shortest vectors of I are outside E_1 , all the nonzero elements of I are outside E_4 (see Figure 2). \square

Remark 3.4. Proposition 3.3 does not show whether the ellipse E_4 exists or not in case the shortest vectors of I are inside the circle E_1 , and I has no nonzero element in the square S_1 (see Figure 3). We will discuss this case in the next sections.

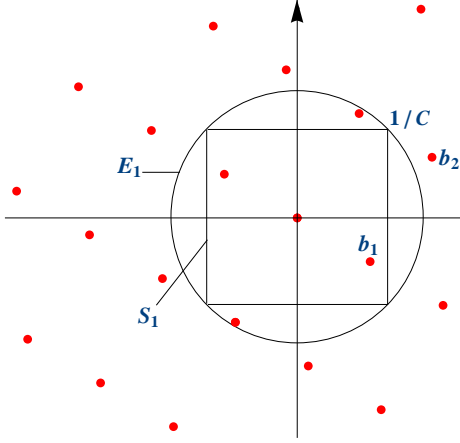


FIGURE 1. The shortest vectors of I are inside the square S_1 .

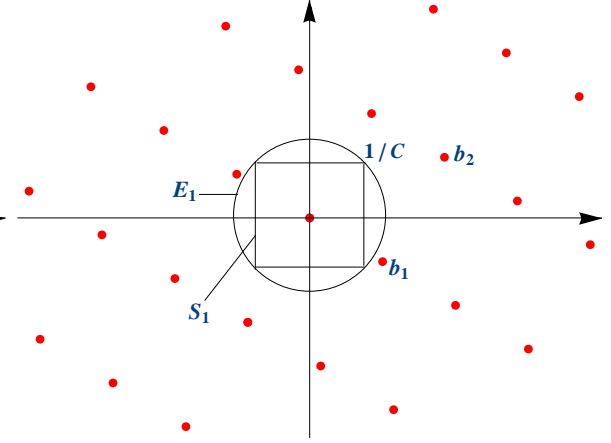


FIGURE 2. The shortest vectors of I are outside the circle E_1 .

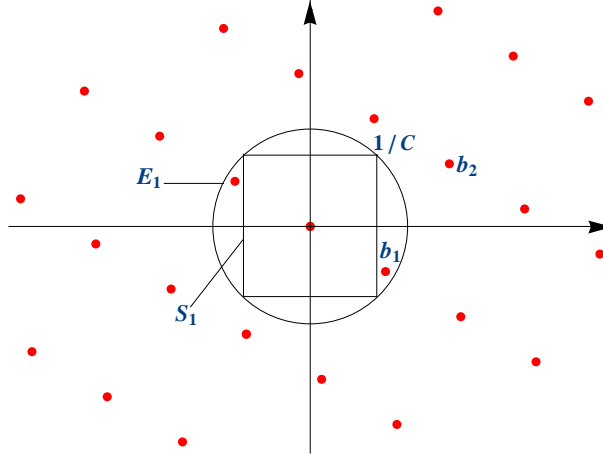


FIGURE 3. The shortest vectors of I are inside E_1 and I has no nonzero element in S_1 .

3.2. Some properties of C -reduced ideals in real quadratic fields. In this section, as mentioned in Remark 3.4, we always assume that I satisfies the conditions (\star) as follows.

- (\star) $\begin{cases} 1) \text{ } 1 \text{ is primitive in } I. \\ 2) \text{ } I \text{ has no nonzero element in the square} \\ S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1/C \text{ and } |x_2| \leq 1/C \text{ and } x_1^2 + x_2^2 < 2/C^2\}. \\ 3) \text{ } \text{A shortest vector } f \text{ of } I \text{ has length } 1/C < \|f\| < \sqrt{2}/C. \end{cases}$

Moreover, by Remark 2.12, we can assume that the vector u in Definition 2.4 has the form $u = (\alpha^{-1}, \alpha) \in (\mathbb{R}_{>0})^2 \subset F_{\mathbb{R}}$ for some $\alpha \in \mathbb{R}_{>0}$.

Let $\{b_1 = (b_{1,1}, b_{1,2}), b_2 = (b_{2,1}, b_{2,2})\}$ be an LLL-basis of I . Then $\|b_1\| = \|f\| < \frac{\sqrt{2}}{C}$. We denote by $\{b_1^*, b_2^*\}$ the Gram-Schmidt orthogonalization of the basis $\{b_1, b_2\}$.

Let

$$G = \{g \in I : \left(g_1^2 - \frac{1}{C^2}\right) \left(g_2^2 - \frac{1}{C^2}\right) < 0 \text{ and } \|g\| < \frac{4}{\pi} C \operatorname{covol}(I)\}.$$

We also set

$$G_1 = \{g \in G : g_1^2 - \frac{1}{C^2} < 0\} \text{ and } G_2 = \{g \in G : g_2^2 - \frac{1}{C^2} < 0\}.$$

So, we obtain $G = G_1 \cup G_2$.

For each $g \in G$, we define

$$B(g) := \left(-\frac{C^2 g_1^2 - 1}{C^2 g_2^2 - 1} \right)^{1/4}.$$

Then denote

$$(3.1) \quad B_{\min} = \begin{cases} \frac{1}{2\sqrt{C}} & \text{if } G_1 = \emptyset \\ \max \{B(g) : g \in G_1\} & \text{if } G_1 \neq \emptyset. \end{cases}$$

$$(3.2) \quad B_{\max} = \begin{cases} 2\sqrt{C} & \text{if } G_2 = \emptyset \\ \min \{B(g) : g \in G_2\} & \text{if } G_2 \neq \emptyset. \end{cases}$$

Let $G' = \{g \in G : B(g) = B_{\max} \text{ or } B(g) = B_{\min}\}$. Then because of assumption (\star) , the vector b_1 is in G . Thus, G' is nonempty.

The most important result in this paper is the following proposition.

Proposition 3.5. *The ideal I is C -reduced if and only if $B_{\min} \leq B_{\max}$.*

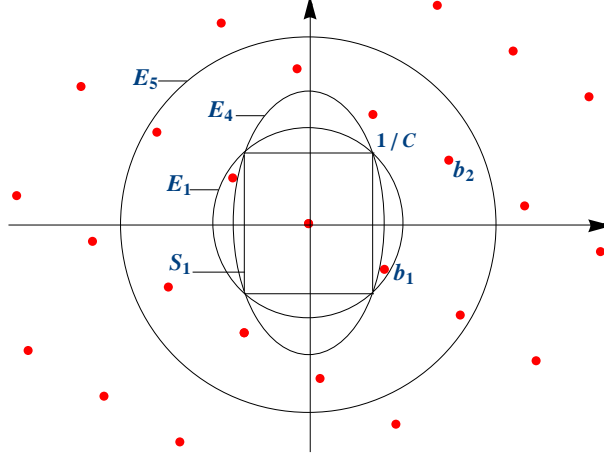
We prove this proposition after proving some results below. First, we establish a property of the ellipses E_4 described in Section 3.1.

Proposition 3.6. *Assume that $E_4 : \frac{X_1^2}{a_1^2} + \frac{X_2^2}{a_2^2} = 1$ with $a_1 > 0$ and $a_2 > 0$ is an ellipse satisfying the conclusion of Proposition 3.2. In other words, E_4 has its center at the origin, passes through the vertices of S_1 and the interior contains no nonzero points of the lattice I . Then:*

- i) *The coefficients a_1 and a_2 are bounded by $\frac{4}{\pi} C \operatorname{covol}(I)$.*
- ii) *E_4 is inside the circle E_5 of radius $\frac{4}{\pi} C \operatorname{covol}(I)$ centered at the origin.*

Proof. Since E_4 passes through the vertex $(1/C, 1/C)$ of S_1 , its coefficients satisfy $a_1 > 1/C$ and $a_2 > 1/C$. We also know that $\operatorname{vol}(E_4) = \pi a_1 a_2$. Hence

$$a_1 = \frac{\operatorname{vol}(E_4)}{\pi a_2} < \frac{1}{\pi} C \operatorname{vol}(E_4).$$

FIGURE 4. Circle E_5 and ellipse E_4 .

In addition, the ellipse E_4 is a symmetric, convex and bounded set whose interior contains no nonzero points of the lattice I , hence it must have volume less than $2^2 \text{covol}(I)$ by Minkowski's theorem. As a consequence,

$$a_1 < \frac{4}{\pi} C \text{covol}(I).$$

By symmetry, we also have this bound for a_2 . Thus, the first statement of the proposition is obtained. The second one follows from the first. \square

We have another equivalent condition to Definition 2.4 as follows.

Proposition 3.7. *The second condition of Definition 2.4 is equivalent to the following: there exists a metric $u \in (\mathbb{R}_{>0})^2$ such that for all $g \in G$, we have $\|1\|_u \leq C\|g\|_u$.*

Proof. Let $g = (g_1, g_2)$ be a nonzero element of I . If $\|g\| \geq (4/\pi)C \text{covol}(I)$ then g is outside the circle E_5 . By Proposition 3.6, g is also outside any ellipse E_4 (see Figure 4). Using this and the equivalent condition of Proposition 3.2, we obtain: a vector u satisfies Definition 2.4 if and only if $\|u\| \leq C\|ug\|$ for all $g \in I \setminus \{0\}$ with $\|g\| < (4/\pi)C \text{covol}(I)$.

On the other hand, if $|g_1| \geq 1/C$ and $|g_2| \geq 1/C$, then g satisfies $\|u\| \leq C\|ug\|$ for any $u \in (\mathbb{R}_{>0})^2$. Therefore, it is sufficient to consider the elements g such that $|g_1| < 1/C$ or $|g_2| < 1/C$ to show the existence of u .

Moreover, I contains no nonzero elements of S_1 , so $g \notin \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1/C \text{ and } |x_2| \leq 1/C \text{ and } x_1^2 + x_2^2 < 2/C^2\}$.

Combining these conditions, we obtain the conclusion. \square

The ideal I with properties (\star) mentioned at the beginning of this section has bounded covolume. Explicitly, we obtain the following.

Proposition 3.8. *The covolume of I is bounded by $2/C$.*

Proof. Since 1 is in I , there exist integers m_1 and m_2 such that $1 = m_1 b_1 + m_2 b_2$. If $m_2 = 0$ then $1 = m_1 b_1$ so $1/m_1 = b_1 \in I$. Because 1 is primitive in I , we must have $m_1 = \pm 1$. Thus $\|b_1\| = \|1\| = \sqrt{2} \geq \sqrt{2}/C$ for any $C \geq 1$. This contradicts the fact that the length of the shortest vectors of I is strictly less than $\sqrt{2}/C$. Hence $m_2 \neq 0$.

We have $\|b_2^*\| \leq \frac{1}{|m_2|} \|1\| \leq \sqrt{2}$. This leads to the following.

$$\text{covol}(I) = \|b_1\| \|b_2^*\| < \frac{\sqrt{2}}{C} \times \sqrt{2} = \frac{2}{C}.$$

□

By this proposition and Proposition 3.6, we obtain the corollary below.

Corollary 3.9. *The coefficients a_1 and a_2 and the radius of the circle E_5 in Proposition 3.6 are bounded by $8/\pi$. In addition, the set G is contained in the finite set $\{g \in I : (g_1^2 - 1/C^2)(g_2^2 - 1/C^2) < 0 \text{ and } \|g\| < 8/\pi\}$.*

For a real quadratic field, the Proposition 2.13 can be restated as below.

Proposition 3.10. *Assume that $u = (\alpha^{-1}, \alpha) \in (\mathbb{R}_{>0})^2$ satisfies the second condition of Definition 2.4. Then $\|u\| \leq 2\sqrt{C}$ and therefore*

$$\frac{1}{2\sqrt{C}} < \alpha < 2\sqrt{C}.$$

Proof. By Proposition 2.13, $\|u\| \leq C\sqrt{2} \text{covol}(I)^{1/2}$. By Proposition 3.8, we have $\text{covol}(I) < \frac{2}{C}$, so $\|u\| \leq 2\sqrt{C}$. Since $\alpha^{-1} < \|u\|$ and $\alpha < \|u\|$, the conclusion follows. □

Proof of Proposition 3.5. Let $u = (\alpha^{-1}, \alpha) \in (\mathbb{R}_{>0})^2$. Then from $\|1\|_u \leq C\|g\|_u$, we get

$$\alpha^4(C^2 g_2^2 - 1) \geq -(C^2 g_1^2 - 1).$$

Thus $\alpha \geq B(g)$ if $g \in G_1$ and $\alpha \leq B(g)$ if $g \in G_2$. As 1 is primitive in I , by Proposition 3.7, the ideal I is C -reduced if and only if it satisfies the following equivalent conditions:
there exists $u \in (\mathbb{R}_{>0})^2$ such that $\|1\|_u \leq C\|g\|_u$ for all $g \in G$,

$$\Leftrightarrow \text{There exist } \alpha \in \mathbb{R}_{>0} \text{ such that } \begin{cases} \alpha \geq B(g) & \text{for all } g \in G_1 \\ \alpha \leq B(g) & \text{for all } g \in G_2 \end{cases}$$

$$\Leftrightarrow \text{There exists } \alpha \in \mathbb{R}_{>0} \text{ such that } \begin{cases} \alpha \geq B_{\min} \\ \alpha \leq B_{\max} \end{cases}$$

$$\Leftrightarrow B_{\max} \geq B_{\min}.$$

The second equivalence comes from Proposition 3.10 and the definition of B_{\min} and B_{\max} . □

Proposition 3.5 and 3.7 motivate a further investigation of properties of the sets G and G' . We first establish a special property of the elements in G .

Proposition 3.11. *If $g = s_1 b_1 + s_2 b_2 \in G$ then $|s_2| \leq 1$.*

Proof. Let $g = s_1b_1 + s_2b_2$ in G . As in the proof of Proposition 3.8, we get $\|b_1\| < \sqrt{2}/C$ and $\|b_2^*\| \leq \sqrt{2}$. By the properties of LLL-reduced bases, $\|b_2\| \leq \sqrt{2}\|b_2^*\| \leq 2$. Therefore,

$$\frac{4C \operatorname{covol}(I)}{\pi} = \frac{4C\|b_1\|\|b_2^*\|}{\pi} < \frac{4\sqrt{2}\|b_2^*\|}{\pi}.$$

Now let g^* be a vector of length equal to the distance from g to the 1-dimensional vector space $\mathbb{R}.b_1$. In other words, $\|g^*\| = d(g, \mathbb{R}.b_1) = |s_2|\|b_2^*\|$. If $|s_2| \geq 2$, then then we would have the following contradiction.

$$\|g\| \geq d(g, \mathbb{R}.b_1) = \|g^*\| \geq 2\|b_2^*\| > \frac{4\sqrt{2}\|b_2^*\|}{\pi} > \frac{4}{\pi}C \operatorname{covol}(I).$$

Thus $|s_2| \leq 1$. □

In the next proposition, we prove that the cardinality of G is bounded by a number that depends only on C but not on I or the number field F .

Lemma 3.12. *The number of vectors in G (up to sign) is less than $17C + 3$.*

Proof. Let $g \in G$. Then $g = s_1b_1 + s_2b_2$ for some integers s_1, s_2 . We have $\|b_1\| \geq 1/C$ and $\|g\| < 8/\pi$ (by Corollary 3.9). This implies that

$$|s_1| \leq \sqrt{2} \left(\frac{3}{2} \right) \frac{\|g\|}{\|b_1\|} < \frac{12\sqrt{2}C}{\pi}$$

[7, Section 12]. By Proposition 3.11, we obtain $|s_2| \leq 1$.

Consequently, the number of elements in G (up to sign) is at most $3 \cdot \left(\frac{12\sqrt{2}C}{\pi} + 1 \right)$, which is less than $17C + 3$. □

The proposition below gives a property of elements in G' .

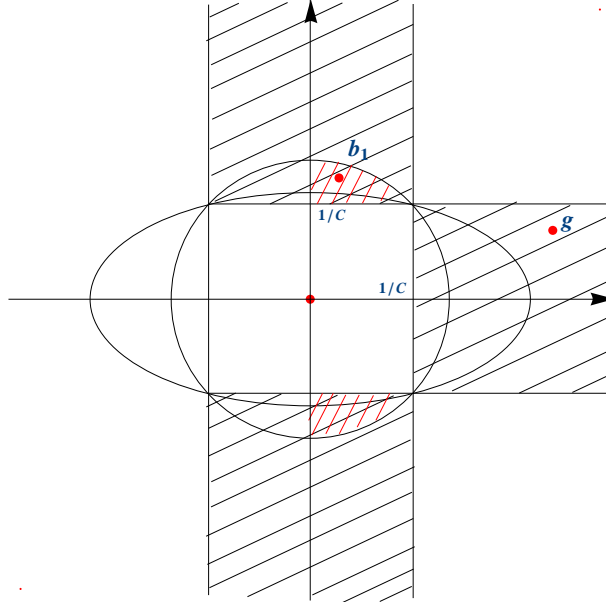
Proposition 3.13. *Let $g = s_1b_1 + b_2 \in G'$. Then:*

- $|s_1| \leq 2$ or
- $s_1 \in \{t_1, t_2\}$ for some integers $t_1 \leq t_2$ in the interval $(-1 - 2C, 1 + 2C)$.

Proof. It is easy to show that $b_1 \in G = G_1 \cup G_2$ since $\|b_1\| \leq (4/\pi)C \operatorname{covol}(I)$. Here, we only prove the proposition for $b_1 \in G_1$, so $0 < b_{11} < 1/C$ and $1/C < |b_{12}| < \sqrt{2}/C$. For $b_1 \in G_2$, it is sufficient to switch b_{11} and b_{12} . In the first case, by definition of B_{\min} , we obtain $B(b_1) \leq B_{\min}$. The element g is in G' , and it belongs to G_1 or G_2 .

If g is in G_1 then $0 < |g_1| < 1/C$ and $|g_2| > 1/C$. Since $g \in G'$ and $B(b_1) \leq B_{\min}$, we also have $B(b_1) \leq B(g)$. If $\|g\| > \sqrt{2}/C$ then $B(b_1) > B(g)$, contradicting the previous inequality. So, $\|g\| \leq \sqrt{2}/C$. With this in mind and the properties of LLL-reduced bases [7, Section 12], we obtain

$$|s_1| \leq \sqrt{2} \left(\frac{3}{2} \right) \frac{\|g\|}{\|b_1\|} < \sqrt{2} \left(\frac{3}{2} \right) \left(\frac{\frac{\sqrt{2}}{C}}{\frac{1}{C}} \right) = 3 \quad \text{so } |s_1| \leq 2.$$

FIGURE 5. b_1 is in the doubly-shaded area and g is in the shaded area.

If g is in G_2 then $|g_1| > 1/C$ and $|g_2| < 1/C$. Since $g = s_1 b_1 + b_2$ and $|g_2| < 1/C$, the value of s_1 is between $\frac{-1/C - b_{22}}{b_{12}}$ and $\frac{1/C - b_{22}}{b_{12}}$. The fact that $0 < |b_{12}| < \sqrt{2}/C$ implies that the distance between these numbers

$$\left| \frac{-1/C - b_{22}}{b_{12}} - \frac{1/C - b_{22}}{b_{12}} \right| = \frac{2}{C|b_{12}|}$$

is in the interval $(\sqrt{2}, 2)$. So, there exist two integers $t_1 \leq t_2$ between these numbers. Moreover, since $1/C < |b_{12}| < \sqrt{2}/C$ and since $|b_{22}| < \|b_2\| \leq 2$ (see the proof of Proposition 3.11), one can easily see that

$$\left| \frac{\pm 1/C - b_{22}}{b_{12}} \right| < 1 + 2C.$$

Thus, the bounds for s_1 are implied, completing the proof. □

4. TEST ALGORITHM FOR REAL QUADRATIC FIELDS

In this section, given $C \geq 1$, we explain an algorithm to test whether a given fractional ideal I is C -reduced for a real quadratic field F in time polynomial in $\log |\Delta_F|$ with Δ_F the discriminant of F .

By Proposition 3.5, if we know B_{min} and B_{max} , then we can show the existence of a metric $u = (\alpha^{-1}, \alpha)$ in Definition 2.4. In this algorithm, we first find all the possible elements of $G' = \{g \in G : B(g) = B_{max} \text{ or } B(g) = B_{min}\}$ and then compute B_{min} and B_{max} . Let $\{b_1, b_2\}$ be an LLL-basis of I and $g = s_1 b_1 + s_2 b_2 \in G'$. Then Proposition

3.11 says that $s_2 = 0$ or $s_2 = \pm 1$. By symmetry, it is sufficient to consider only the case $s_2 \in \{0, 1\}$.

- If $s_2 = 0$ then $g = b_1$.
- If $s_2 = 1$ then $g = s_1 b_1 + b_2$. By Proposition 3.13, there are five possible values for s_1 in the interval $[-2, 2]$ and two possible values t_1, t_2 (with $t_1 \leq t_2$) of s_1 either between $\frac{-1/C-b_{22}}{b_{12}}$ and $\frac{1/C-b_{22}}{b_{12}}$ or between $\frac{-1/C-b_{21}}{b_{11}}$ and $\frac{1/C-b_{21}}{b_{11}}$. This proposition also shows that the coefficients s_1 have absolute values less than $1 + 2C$.

Furthermore, by Proposition 3.10, we have $\frac{1}{2\sqrt{C}} < \alpha < 2\sqrt{C}$ and so $\frac{1}{16C^2} < B(g)^4 < 16C^2$ for all $g \in G$. In other words:

$$(\star\star) \begin{cases} \text{If } |g_2| < 1/C \text{ then} \\ \quad |g_1|^2 + 16C^2|g_2|^2 < 16 + \frac{1}{C^2} \text{ and } |g_2|^2 + 16C^2|g_1|^2 > 16 + \frac{1}{C^2}. \\ \text{If } |g_2| > 1/C \text{ then} \\ \quad |g_2|^2 + 16C^2|g_2|^2 > 16 + \frac{1}{C^2} \text{ and } |g_2|^2 + 16C^2|g_1|^2 > 16 + \frac{1}{C^2}. \end{cases}$$

The statements in $(\star\star)$ can be applied to eliminate some elements g which are not in G' without having to compute $B(g)$.

Let $C \geq 1$ and let I be a fractional ideal of F . Assume that an LLL-reduced basis $\{b_1, b_2\}$ of I is also given and change the sign if necessary to have the first component of $b_1 = (b_{11}, b_{12}) \in F_{\mathbb{R}}$ positive. In Remark 2.9, we assume that the coordinates of b_1 and b_2 have at most $O((\log |\Delta_F|)^a)$ digits for some integer $a > 0$.

We have the following algorithm to test whether I is C -reduced in time polynomial in $\log |\Delta_F|$.

Algorithm 4.1.

1. Check if $1 \in I$ and $N(I)^{-1} < C^2 \sqrt{|\Delta_F|}$ or not.
2. Test whether or not $1 \in I$ is primitive.
3. Check whether there is no nonzero element of I in $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1/C \text{ and } |x_2| \leq 1/C \text{ and } x_1^2 + x_2^2 < 2/C^2\}$.
4. If $\|b_1\| \geq \sqrt{2}/C$ then I is C -reduced.
If not, then find all possible elements of G' .
 - If $0 < b_{11} < 1/C$ and $1/C < |b_{12}| < \sqrt{2}/C$ then compute the integers $t_1 \leq t_2$ which are between $\frac{-1/C-b_{22}}{b_{12}}$ and $\frac{1/C-b_{22}}{b_{12}}$.
 - If $1/C < b_{11} < \sqrt{2}/C$ and $0 < |b_{12}| < 1/C$ then compute the integers $t_1 \leq t_2$ which are between $\frac{-1/C-b_{21}}{b_{11}}$ and $\frac{1/C-b_{21}}{b_{11}}$.
 Let $G_3 = \{b_1, t_1 b_1 + b_2, t_2 b_1 + b_2, s_1 b_1 + b_2 \text{ with } |s_1| \leq 2\}$.
5. Remove from G_3 all elements which do not satisfy $(\star\star)$.
6. Compute $B(g)$ for all $g \in G_3$, and then B_{\max} and B_{\min} .

If $B_{\min} \leq B_{\max}$ then I is C -reduced. If not, then I is not C -reduced.

Step 3 of Algorithm 4.1 is done in a similar way to testing the minimality of 1 was done (cf.[10, Algorithm 10.3]) but here 1 is replaced by $\frac{1}{C}$. In fact, we have the lemma below.

Lemma 4.2. *Step 3 of Algorithm 4.1 can be done by checking at most six short vectors of the lattice I .*

Proof. If b_1 is in S_1 then I is not C -reduced. Otherwise, $\|b_1\| > \frac{1}{C}$. Assume that $g = s_1b_1 + s_2b_2$ is in S_1 . Then g has length $\|g\| < \frac{\sqrt{2}}{C}$.

Since $\{b_1, b_2\}$ is an LLL-reduced basis of I , the coefficients s_1 and s_2 are bounded as

$$|s_1| \leq \sqrt{2} \left(\frac{3}{2} \right) \frac{\|g\|}{\|b_1\|} < \sqrt{2} \left(\frac{3}{2} \right) \left(\frac{\frac{\sqrt{2}}{C}}{\frac{1}{C}} \right) = 3$$

and

$$|s_2| \leq \sqrt{2} \frac{\|g\|}{\|b_1\|} < \sqrt{2} \left(\frac{\frac{\sqrt{2}}{C}}{\frac{1}{C}} \right) = 2$$

[7, Section 12]. Therefore, the elements of I which are in S_1 have the form $g = s_1b_1 + s_2b_2$ with $|s_1| \leq 2$ and $|s_2| \leq 1$. By symmetry, it is sufficient to test at most six short elements of I . \square

Proposition 4.3. *Algorithm 4.1 runs in time polynomial in $\log |\Delta_F|$.*

Proof. The first step can be done in polynomial time in $\log |\Delta_F|$ by Lemma 2.10. An LLL-reduced basis of I can be computed in time polynomial in $\log |\Delta_F|$ and Step 2 can be done in time polynomial in $\log |\Delta_F|$ (see Lemma 2.11 in Section 2). In Step 3, by Lemma 4.2, it is sufficient to check few short vectors of I which have length bounded by $\frac{\sqrt{2}}{C}$. Step 4 can be done by finding 2 integer numbers t_1, t_2 which are in the interval $[-1 - 2C, 1 + 2C]$. In Step 6, the bounds $B(g)$ are between $\frac{1}{2\sqrt{C}}$ and $2\sqrt{C}$. Overall, this algorithm runs in time polynomial in $\log |\Delta_F|$. \square

5. A COUNTEREXAMPLE

By Lemma 2.10, 2.11 and 4.2, the first three steps of Algorithm 4.1 can be done in time polynomial in $\log |\Delta_F|$. Essentially, the last three steps require finding all elements of I in a certain subset G (see Proposition 3.7 and Lemma 5.1). Therefore, the complexity of this algorithm is proportional to the cardinality of G .

For real quadratic fields, the task can be reduced to finding all elements of the subset G' of G by Proposition 3.5. Since Proposition 3.11 and 3.13 say that G' have few elements and it is easy to compute them, Algorithm 4.1 works well, i.e., it runs in time polynomial in $\log |\Delta_F|$.

However, for a number field of degree at least 3, the set G may have many elements, and we currently do not know how to reduce G to a smaller subset. Therefore, an algorithm similar to Algorithm 4.1 would be inefficient. In other words, in bad cases, the complexity of Step 4–6 of Algorithm 4.1 may reach $|\Delta_F|^a$ for some $a > 0$. In this

section, we provide an example of a real cubic field F with large discriminant Δ_F for which G has at least $|\Delta_F|^{1/4}$ elements.

Since F is a real cubic field, we have $F_{\mathbb{R}} \cong \mathbb{R}^3$. Let I be a fractional ideal of F . Then we identify each element $g \in I$ with its image $(\sigma(g))_{\sigma} = (g_1, g_2, g_3) \in F_{\mathbb{R}} \cong \mathbb{R}^3$.

We set

$$\delta(I, C) = \frac{6}{\pi} C^2 \text{covol}(I)$$

and let

$$S_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| \leq 1/C, 1 \leq i \leq 3 \text{ and } x_1^2 + x_2^2 + x_3^2 < 3/C^2\},$$

$$G = \{g = (g_1, g_2, g_3) \in I : \|g\| < \delta(I, C) \text{ and}$$

$$\text{there exists } i \text{ such that } |g_i| < 1/C\}.$$

Let E_1 be the sphere centered at the origin of radius $\frac{\sqrt{3}}{C}$. As condition (\star) for the quadratic case (see Remark 3.4), we assume that 1 is primitive in I and I contains no nonzero element of S_1 but the shortest vectors of I are inside E_1 .

Proposition 3.2 and 3.6 for the quadratic case can be naturally generalized to a real cubic field. Similar to Proposition 3.7, we have the following result.

Lemma 5.1. *The second condition of Definition 2.4 is equivalent to: there exists a metric $u \in (\mathbb{R}_{>0})^3$ such that $\|1\|_u \leq C\|g\|_u$ for all $g \in G$.*

Let $\{b_1, b_2, b_3\}$ be an LLL-basis of I . We give an example with $C = 1$.

5.1. An example. Let $P(X) = 10000000019X^3 + 10218400019X^2 - 8813199073X - 4923977196$ be an irreducible polynomial with a root β and $F = \mathbb{Q}(\beta)$. Then F is a real cubic field with discriminant

$$\Delta_F = 70862499223222398531211367826392679055149 > 7 \cdot 10^{40}.$$

Denote by O_F the ring of integers of F . Let $I = O_F + O_F\beta$. Then the fractional ideal I has the properties that:

- 1 is primitive in I .
- I has no nonzero element in the cube S_1 .
- b_1 is inside E_1 so are the shortest vectors of I .
- The covolume of I is greater than $1.6 \cdot |\Delta_F|^{1/4}$.

The cardinality of G is at least $1.7 \cdot 10^{10} > |\Delta_F|^{1/4}$.

5.2. How to find the above example. We construct a real cubic field F with a fractional ideal I satisfying the conditions of Section 5.1.

Let $C \geq 1$. Assume that $F = \mathbb{Q}(\beta)$ for some β of length $\|\beta\| < \sqrt{3}/C$ and outside the cube S_1 . Let O_F be the ring of integers of F . Suppose that $I = O_F + O_F\beta$. Then the shortest vectors of I have length at most $\|\beta\| < \sqrt{3}/C$.

Denote by $P(X) = aX^3 + bX^2 + cX + d \in \mathbb{Z}[X]$ with $\gcd(a, b, c, d) = 1$ and $a > 0$ an irreducible polynomial that has a root β . Let

$$R = \mathbb{Z} \oplus \mathbb{Z}(a\beta) \oplus \mathbb{Z}(a\beta^2 + b\beta).$$

Then R is a multiplier ring. Hence it is an order of F [4, Section 12.6].

Denote by $\beta_1 = \beta, \beta_2$ and β_3 the roots of $P(X)$. We can easily choose $P(X)$ such that $O_F = R$. This can be done by using the lemma below.

Lemma 5.2. *If the discriminant of $P(X)$ is squarefree then $O_F = R$.*

Proof. The discriminant of $P(X)$ is $\text{disc}(P) = a^4 \prod_{i < j} (\beta_i - \beta_j)^2$ [3, Proposition 3.3.5]. By computing the discriminant of R , we can easily see that it is equal to $\text{disc}(P)$. The result follows since $[O_F : R]^2 \mid \text{disc}(P)$. \square

Lemma 5.3. *If $O_F = R$ then $N(I^{-1}) = a$.*

Proof. Since $O_F = R = \mathbb{Z} \oplus \mathbb{Z}(a\beta) \oplus \mathbb{Z}(a\beta^2 + b\beta)$ and $I = O_F + O_F\beta$, it is easy to see that $I = \mathbb{Z} \oplus \mathbb{Z}\beta \oplus \mathbb{Z}(a\beta^2)$. It leads to $N(I^{-1}) = [I : O_F] = a$ and the lemma is proved. \square

The next lemma says that a can be chosen such that 1 is primitive in I .

Lemma 5.4. *If a is a prime number then 1 is primitive in I .*

Proof. If there is an integer $d \geq 2$ such that $1/d \in I$, then $d^3 = N(d)N(I^{-1}) = a$, impossible since a is a prime. Thus, 1 is primitive in I . \square

Let $\{b_1 = (b_{11}, b_{12}, b_{13}), b_2 = (b_{21}, b_{22}, b_{23}), b_3 = (b_{31}, b_{32}, b_{33})\} \subset \mathbb{R}^3 \subset F_{\mathbb{R}}$ and $\{b_1^*, b_2^*, b_3^*\}$ the Gram-Schmidt orthogonalization of this basis. We have the following result, crucial to obtaining the example of Section 5.1.

Proposition 5.5. *Let $C \geq 1$. Assume that:*

- 1 is primitive in I .
- I has no nonzero elements in the cube S_1 .
- $\|b_1\| < \sqrt{3}/C$.
- $\text{covol}(I) \geq 10$.

Then the cardinality of G is at least $\frac{2}{3}C^2 \text{covol}(I)$.

Proof. As I has no nonzero element in S_1 , there is some coordinate b_{1j} with $1 \leq j \leq 3$ of b_1 such that $|b_{1j}| \geq 1/C$. Let $g = s_1 b_1 + s_2 b_2 = (g_1, g_2, g_3)$. We show that if $|s_2| \leq \frac{1}{3}C^2 \text{covol}(I)$ and if s_1 is between two numbers $\frac{1}{b_{1j}}(1/C - s_2 b_{2j})$ and $\frac{1}{b_{1j}}(-1/C - s_2 b_{2j})$, then g is in G .

We know that $\|b_1\| < \sqrt{3}/C$, hence $|b_{1j}| < \sqrt{3}/C$. This means that for each s_2 , the distance between $\frac{1}{b_{1j}}(1/C - s_2 b_{2j})$ and $\frac{1}{b_{1j}}(-1/C - s_2 b_{2j})$ is greater than $2/\sqrt{3} > 1$. Therefore there is at least one integer s_1 between them.

The bound for s_1 implies that $|g_j| < \frac{1}{C}$. To prove that $g \in G$, it is sufficient to prove that $\|g\| < \delta(I, C)$.

We first show that $\|b_2\| \leq \sqrt{3}$. Since 1 is in I , there exist integers m_1, m_2 and m_3 such that $1 = m_1 b_1 + m_2 b_2 + m_3 b_3$. If $m_3 = m_2 = 0$ then $1 = m_1 b_1$. It follows that $1/m_1 = b_1 \in I$. Since 1 is primitive, we must have $m_1 = \pm 1$. So, $\|b_1\| = \|1\| = \sqrt{3} \geq \frac{\sqrt{3}}{C}$ for any $C \geq 1$. This contradicts $\|b_1\| < \sqrt{3}/C$. As a result, $m_3 \neq 0$ or $m_2 \neq 0$. If

$m_3 \neq 0$, then $\|b_3^*\| \leq \frac{1}{m_3}\|1\| \leq \sqrt{3}$. By the properties of LLL-reduced bases [7, Section 12], we have $\|b_2^*\| \leq \sqrt{2}\|b_3^*\| \leq \sqrt{6}$. Then

$$\text{covol}(I) = \|b_1\| \|b_2^*\| \|b_3^*\| < \frac{\sqrt{3}}{C} \cdot \sqrt{6} \cdot \sqrt{3} = \frac{3\sqrt{6}}{C},$$

contrary to the assumption that $\text{covol}(I) \geq 10$. Hence, $m_3 = 0$ and $m_2 \neq 0$. Consequently, $\|b_2^*\| \leq \frac{1}{|m_2|}\|1\| \leq \sqrt{3}$.

Next, we prove that $\|b_2\| \leq \frac{\sqrt{15}}{2}$. Indeed, denoting $\mu = \langle b_2, b_1 \rangle / \langle b_1, b_1 \rangle$, by the properties of LLL-reduced bases we have $|\mu| \leq \frac{1}{2}$ and $b_2 = b_2^* + \mu b_1$ [7, Section 12]. It follows that

$$\|b_2\|^2 = \|b_2^*\|^2 + \mu^2 \|b_1\|^2 < 3 + \frac{1}{4} \frac{3}{C^2} \leq \frac{15}{4}.$$

Now, since $|b_{1j}| \geq \frac{1}{C}$ and $|b_{2j}| \leq \|b_2\| \leq \frac{\sqrt{15}}{2}$, the two numbers $\frac{1}{b_{1j}}(1/C - s_2 b_{2j})$ and $\frac{1}{b_{1j}}(-1/C - s_2 b_{2j})$ are in the interval

$$\left[-\left(1 + \frac{\sqrt{15}}{2}|s_2|\right)C, \left(1 + \frac{\sqrt{15}}{2}|s_2|\right)C \right]$$

and so is s_1 . Therefore

$$\begin{aligned} \|g\|^2 &= \|(s_1 + \mu s_2)b_1 + s_2 b_2^*\|^2 = (s_1 + \mu s_2)^2 \|b_1\|^2 + |s_2|^2 \|b_2\|^2 \\ &< \left(\left(1 + \frac{\sqrt{15}}{2}|s_2|\right)C + \frac{1}{2}|s_2| \right)^2 \frac{3}{C^2} + 3s_2^2 \\ &\leq 3 \left(1 + \frac{1 + \sqrt{15}}{2}|s_2| \right)^2 + 3s_2^2 < [\delta(I, C)]^2 \end{aligned}$$

since $|s_2| \leq \frac{1}{3}C^2 \text{covol}(I)$ and $\text{covol}(I) \geq 10$.

We have shown that $g = s_1 b_1 + s_2 b_2 \in G$ for all $(s_1, s_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ with $|s_2| \leq \frac{1}{3}C^2 \text{covol}(I)$ and s_1 between $\frac{1}{b_{1j}}(1/C - s_2 b_{2j})$ and $\frac{1}{b_{1j}}(-1/C - s_2 b_{2j})$. Furthermore, if $g \in G$, then $-g \in G$. Thus, G has at least $[2 \cdot \frac{1}{3}C^2 \text{covol}(I) = \frac{2}{3}C^2 \text{covol}(I)]$ elements. \square

Corollary 5.6. *With the assumptions in Proposition 5.5, the set G contains more than $\gamma C^2 |\Delta_F|^{1/4}$ elements for some constant γ depending on the roots $\beta_1, \beta_2, \beta_3$ of P .*

Proof. By choosing P such that $O_F = R$, we have

$$|\Delta_F| = \text{disc}(R) = \text{disc}(P) = a^4 \prod_{i < j} (\beta_i - \beta_j)^2.$$

Hence

$$a = \frac{1}{\gamma} |\Delta_F|^{1/4} \text{ with } \gamma = \left(\prod_{i < j} (\beta_i - \beta_j)^2 \right)^{1/4}.$$

Consequently,

$$\text{covol}(I) = \frac{\sqrt{|\Delta_F|}}{N(I^{-1})} = \frac{|\Delta_F|^{1/2}}{a} = \gamma |\Delta_F|^{1/4}$$

and the result follows from Proposition 5.5. \square

Remark 5.7. *Almost all the lattices I constructed this way have no nonzero element in the cube S_1 as we may expect. Indeed, any element $g = s_1b_1 + s_2b_2 + s_3b_3 \in I \cap S_1$ has length at most $\sqrt{3}/C$. So, we can bound for the coefficients s_1, s_2, s_3 as follows [7, Section 12].*

$$|s_1| \leq 2 \left(\frac{3}{2}\right)^2 \frac{\|g\|}{\|b_1\|}, \quad |s_2| \leq 2 \left(\frac{3}{2}\right) \frac{\|g\|}{\|b_2^*\|}, \quad |s_3| \leq 2 \frac{\|g\|}{\|b_3^*\|}.$$

Therefore, the cardinality of $I \cap S_1$ is bounded by

$$\frac{1}{\text{covol}(I)} \cdot \left(\frac{\sqrt{3}}{C}\right)^3 \cdot (\text{a constant})$$

[7, Section 12]. Since the covolume of I is very large, this number is very small. So, usually we can get I without any nonzero elements in S_1 .

From the idea above, some examples like the one in 5.1 can be produced as follows.

- First choose the discriminant $|\Delta_F|$ of F such that $|\Delta_F| > 10^4$ (to make sure that $\text{covol}(I) \geq 10$).
- Choose a prime number $a \approx |\Delta_F|^{1/4}$ (such that 1 is primitive in I).
- Chose a real vector $(\beta_1, \beta_2, \beta_3)$ outside S_1 and such that

$$\frac{1}{C^2} < \beta_1^2 + \beta_2^2 + \beta_3^2 < \frac{3}{C^2}.$$

- Find the polynomial $P(X) = aX^3 + bX^2 + cX + d \in \mathbb{Z}[X]$ of the form $a(X - \beta_1)(X - \beta_2)(X - \beta_3)$ (this can be done by using the function `round` in `pari-gp`). Then check whether $P(X)$ is irreducible. Check if $\text{disc}(P)$ is squarefree. If not then change β_i until it is. Now $O_F = R$.
- Let $I = O_F + O_F\beta$. Compute an LLL-reduced basis $\{b_1, b_2, b_3\}$ of I and check if $\|b_1\| < \sqrt{3}/C$.
- Test whether I does not have any nonzero element in S_1 .

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DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, AALTO UNIVERSITY SCHOOL OF SCIENCE, OTAKAARI 1, 02150 ESPOO, FINLAND.

E-mail address: hatran1104@gmail.com